

# Central Bag Boundary Conditions on the Ball

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## Abstract

Local boundary conditions for spinor fields are expressed in terms of a 1-parameter family of boundary operators, and find applications ranging from (supersymmetric) quantum cosmology to the bag model in quantum chromodynamics. The present paper proves that, for massless spinor fields on the Euclidean ball in dimensions  $d = 2, 4, 6$ , the resulting  $\zeta(0)$  value is independent of such a  $\theta$  parameter, while the various heat-kernel coefficients exhibit a  $\theta$ -dependence which is eventually expressed in a simple way through hyperbolic functions and their integer powers.

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## I. INTRODUCTION

The choice of boundary conditions in the theories of fundamental interactions has always attracted the interest of theoretical physicists, not only as a part of the general programme aimed at deriving the basic equations of physics from a few guiding principles [1–8], but also as a tool for studying concrete problems in quantum field theory and global analysis [9–11].

In particular, we are here interested in studying local boundary conditions for massless spin- $\frac{1}{2}$  fields, whose main motivations may be summarized as follows [12,13].

(i) The Breitenlohner–Freedman–Hawking [14,15] boundary conditions for gauged supergravity theories in anti-de Sitter space are local and are expressed, for spin- $\frac{1}{2}$  fields, in terms of a projection operator. The rigid supersymmetry transformations between massless linearized fields of different spins map classical solutions of the linearized field equations, subject to such boundary conditions at infinity, to classical solutions for an adjacent spin, subject to the same family of boundary conditions at infinity [12].

(ii) In simple supergravity the spatial tetrad and a projection formed from the spatial components of the spin- $\frac{3}{2}$  potential transform into each other under half of the local supersymmetry transformations at the boundary [16]. The supergravity action functional can also be made invariant under this class of local supersymmetry transformations. On considering the extension to supergravity models based on the group  $O(N)$ , the supersymmetry transformation laws show that, *for spin- $\frac{1}{2}$  fields only*, the same projector should be specified on the boundary as in the Breitenlohner–Freedman–Hawking case.

(iii) The work in Ref. [13] has shown that, instead of quantizing gauge theories on a sphere or on a torus one can quantize them in an even-dimensional Euclidean bag and impose  $SU_A(N_f)$ -breaking boundary conditions to trigger a chiral symmetry breaking. On investigating how the various correlators depend on the parameter  $\theta$  characterizing the boundary conditions one then finds that bag boundary conditions are a substitute for small quark masses [13].

More precisely, in theories of Euclidean bags, chiral symmetry breaking is triggered by imposing the boundary conditions [13,17]

$$0 = \pi_- \psi|_{\partial M} = \frac{1}{2} \left( 1 + i e^{\theta \gamma^5} \gamma^5 \gamma^m \right) \psi|_{\partial M} \quad (1.1)$$

on the spinor field  $\psi$ . Here we focus on the  $d$ -dimensional Euclidean ball, which is the portion of flat  $d$ -dimensional Euclidean space bounded by the  $S^{d-1}$  sphere. The eigenspinors of the Dirac operator on the ball have the form [18]

$$\psi_{\pm}^{(+)} = \frac{C}{r^{(d-2)/2}} \begin{pmatrix} i J_{n+d/2}(kr) Z_+^{(n)}(\Omega) \\ \varepsilon J_{n+(d-2)/2}(kr) Z_+^{(n)}(\Omega) \end{pmatrix}, \quad (1.2)$$

$$\psi_{\pm}^{(-)} = \frac{C}{r^{(d-2)/2}} \begin{pmatrix} \varepsilon J_{n+(d-2)/2}(kr) Z_-^{(n)}(\Omega) \\ i J_{n+d/2}(kr) Z_-^{(n)}(\Omega) \end{pmatrix}, \quad (1.3)$$

where  $C$  is a normalization constant,  $\varepsilon \equiv \pm 1$ ,  $n = 0, 1, 2, \dots, \infty$ , and  $Z_{\pm}^{(n)}(\Omega)$  are the spinor modes on the sphere [19]. In Eq. (1.1), the boundary operator reduces to the matrix

$$\frac{1}{2} \begin{pmatrix} 1 & -i e^{\theta} \\ i e^{-\theta} & 1 \end{pmatrix},$$

and its application to (1.2) and (1.3) yields the eigenvalue condition [20]

$$J_{n+d/2}(k) - \varepsilon e^{\theta} J_{n+d/2-1}(k) = 0 \quad (1.4)$$

for  $\psi_{\pm}^{(+)}$ , and

$$J_{n+d/2}(k) + \varepsilon e^{-\theta} J_{n+d/2-1}(k) = 0 \quad (1.5)$$

for  $\psi_{\pm}^{(-)}$ , where  $r$  has been set to 1 for convenience [3]. By eigenvalue condition we mean the equation obeyed by the eigenvalues by virtue of the boundary conditions, which yields them only implicitly [3]. Equations (1.4) and (1.5) lead eventually to the eigenvalue condition in non-linear form, i.e.

$$J_{n+d/2-1}^2(k) - e^{-2\theta} J_{n+d/2}^2(k) = 0, \quad (1.6)$$

$$J_{n+d/2-1}^2(k) - e^{2\theta} J_{n+d/2}^2(k) = 0. \quad (1.7)$$

Of course, it is enough to deal with one of these equations, while the contributions from the other follow by replacing  $\theta$  with  $-\theta$ .

Recently, in  $d = 2$  dimensions, the spectral asymmetry following from the boundary conditions (1.1) was considered in [21]. Asymmetry properties are encoded in the eta function which was analyzed using contour integral methods, see e.g. [11].

Instead, we study heat-kernel asymptotics for the squared Dirac operator on the  $d$ -ball with eigenvalue conditions (1.6) and (1.7) which is related to an analysis of the zeta function. Strictly, one can actually obtain two second-order operators of Laplace type out of the Euclidean Dirac operator  $D$ , i.e.

$$P_1 \equiv DD^\dagger \text{ and } P_2 \equiv D^\dagger D,$$

where  $D^\dagger$  denotes the (formal) adjoint of  $D$ . The existence of both  $P_1$  and  $P_2$  is crucial for index theory [5], and by taking into account both (1.6) and (1.7) we correctly take care of this (see Ref. [12] for the mode-by-mode version of  $P_1$  and  $P_2$  on the 4-ball). To be self-contained, recall that, given the second-order elliptic operator  $P$ , the heat kernel can be defined as the solution, for  $\tau > 0$ , of the associated heat equation

$$\left(\frac{\partial}{\partial\tau} + P\right)U(x, y; \tau) = 0, \tag{1.8}$$

subject to the initial condition ( $(M, g)$  being the background geometry)

$$\lim_{\tau \rightarrow 0} \int_M U(x, y; \tau) \varphi(y) \sqrt{\det g} \, dy = \varphi(x), \tag{1.9}$$

and to suitable boundary conditions

$$[\mathcal{B}U(x, y; \tau)]_{\partial M} = 0, \tag{1.10}$$

which preserve ellipticity and lead to self-adjointness of the boundary-value problem [9–11]. The functional (or  $L^2$ ) trace of the heat kernel is obtained by considering the heat-kernel diagonal  $U(x, x; \tau)$ , taking its fibre trace  $\text{Tr}_V U(x, x; \tau)$  (since  $U(x, y; \tau)$  carries (implicit) group indices in the case of gauge theories), and integrating such a fibre trace over  $M$ , i.e.

$$\mathrm{Tr}_{L^2} e^{-\tau P} = \int_M \mathrm{Tr}_V U(x, x; \tau) \sqrt{\det g} \, dx. \quad (1.11)$$

The asymptotic expansion we are interested in holds for  $\tau \rightarrow 0^+$  and has the form [9]

$$\mathrm{Tr}_{L^2}(e^{-\tau P}) \sim \tau^{-d/2} \sum_{n=0}^{\infty} \tau^{n/2} a_{n/2}(P, \mathcal{B}), \quad (1.12)$$

where the heat-kernel coefficients  $a_{n/2}(P, \mathcal{B})$  are said to describe the global (integrated) asymptotics and consist of an interior part  $c_{n/2}(P)$  and a boundary part  $b_{n/2}(P, \mathcal{B})$ , i.e.

$$a_{n/2}(P, \mathcal{B}) = c_{n/2}(P) + b_{n/2}(P, \mathcal{B}). \quad (1.13)$$

At a deeper level, we might introduce a smearing function and consider instead the  $L^2$  trace of  $f e^{-\tau P}$ , with  $f$  a smooth function on  $M$ . This takes into account the distributional behaviour of the heat kernel from the point of view of invariance theory (here “invariance” refers to the invariants of the orthogonal group, which determine completely the functional form of  $a_{n/2}(P, \mathcal{B})$  [9]). However, mode-by-mode calculations like the ones we are going to consider can be performed without exploiting the introduction of  $f$ , and hence we limit ourselves to using Eqs. (1.11)–(1.13). Section II describes the  $\zeta$ -function algorithm in Ref. [22] on the Euclidean  $d$ -ball [23], and Sec. III generalizes the work in Ref. [12] by showing that, on the 4-ball, non-vanishing values of  $\theta$  in Eqs. (1.6) and (1.7) do not affect the conformal anomaly. The hardest part of our analysis is then presented in Secs. IV and V, where heat-kernel coefficients are studied for arbitrary dimension  $d$ , with several explicit formulae in  $d = 2, 4, 6$ . Concluding remarks and open problems are described in Sec. VI, while relevant details can be found in the Appendix.

## II. THE MOSS ALGORITHM FOR THE $D$ -BALL

The starting point in our investigation of the eigenvalue condition (1.7) for the purpose of heat-kernel asymptotics is the use of the  $\zeta$ -function at large  $x$ , which was first described in Ref. [22] with application to 4-dimensional background geometries. However, since we are interested in the Euclidean  $d$ -ball, we put no restriction on the dimension of  $M$ , denoted

by  $d$  as in Sec. I, and we follow the general procedure as outlined by Dowker [23]. First we point out that on replacing the eigenvalues  $\lambda_n$  of  $P$  by  $\lambda_n + x^2$  ( $x$  being a large real parameter), one has the  $\zeta$ -function at large  $x$  in the form

$$\zeta(s, x^2) \equiv \sum_n (\lambda_n + x^2)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} U_x(\tau) d\tau, \quad (2.1)$$

having defined the integrated heat kernel (or functional trace of the heat kernel at large  $x$ ) as

$$U_x(\tau) \equiv \sum_n e^{-(\lambda_n + x^2)\tau} = e^{-x^2\tau} U(\tau). \quad (2.2)$$

By virtue of the asymptotic expansion already encountered in the Introduction, i.e.

$$U(\tau) \equiv \sum_n e^{-\lambda_n \tau} \sim \sum_{n=0}^\infty a_{n/2} \tau^{\frac{(n-d)}{2}} \text{ as } \tau \rightarrow 0^+, \quad (2.3)$$

we therefore find

$$\zeta(s, x^2) \sim \frac{1}{\Gamma(s)} \sum_{n=0}^\infty a_{n/2} I(x; s, n, d), \quad (2.4)$$

having defined

$$I(x; s, n, d) \equiv \int_0^\infty \tau^{s-1+\frac{(n-d)}{2}} e^{-x^2\tau} d\tau. \quad (2.5)$$

Now we distinguish two cases, depending on whether  $d$  is even or odd. In the former, we consider  $s = \bar{s}$  such that  $\bar{s} - 1 - \frac{d}{2} = 0$ , i.e.  $\bar{s} \equiv 1 + \frac{d}{2}$  which implies (on defining  $\tau x^2 \equiv z$ )

$$I(x; \bar{s}, n, d) = \int_0^\infty \tau^{\frac{n}{2}} e^{-x^2\tau} d\tau = x^{-n-2} \Gamma\left(1 + \frac{n}{2}\right), \quad (2.6)$$

and hence yields, for  $d = 2k, k = 0, 1, 2, \dots$

$$\zeta\left(1 + \frac{d}{2}, x^2\right) \sim \sum_{n=0}^\infty a_{n/2} \frac{\Gamma\left(1 + \frac{n}{2}\right)}{\Gamma\left(1 + \frac{d}{2}\right)} x^{-n-2}. \quad (2.7)$$

In the latter, we consider  $s = \tilde{s}$  such that

$$s = 1 + \frac{(d-1)}{2} \equiv \tilde{s},$$

which implies

$$I(x; \tilde{s}, n, d) = \int_0^\infty \tau^{\frac{n-1}{2}} e^{-x^2 \tau} d\tau = x^{-n-1} \Gamma\left(\frac{1+n}{2}\right). \quad (2.8)$$

On the other hand, since the function expressing the eigenvalue condition (1.7) admits a canonical product representation (see Appendix), one can prove, on setting  $\nu \equiv n + \frac{d}{2}$  for  $d$  even, the identity

$$\begin{aligned} \Gamma\left(1 + \frac{d}{2}\right) \zeta\left(1 + \frac{d}{2}, x^2\right) &= (-1)^{\frac{d}{2}} \sum_{n=0}^\infty 2^{\frac{d}{2}-1} \binom{d+n-2}{n} \left(\frac{1}{2x} \frac{d}{dx}\right)^{1+\frac{d}{2}} \\ &\times \log\left[(ix)^{-2(\nu-1)} (J_{\nu-1}^2(ix) - e^{2\theta} J_\nu^2(ix))\right], \end{aligned} \quad (2.9)$$

where  $2^{\frac{d}{2}}$  is the dimension  $d_s$  of spinor space, and  $\deg(n) = \frac{1}{2}d_s \binom{d+n-2}{n}$  is the degeneracy associated with the implicit eigenvalue condition (1.7). Thus, the heat-kernel coefficient  $a_{\frac{l}{2}}$  is equal to  $\frac{1}{\Gamma(1+\frac{l}{2})}$  (respectively  $\frac{1}{\Gamma(\frac{1+l}{2})}$ ) times the coefficient of  $x^{-l-2}$  (respectively  $x^{-l-1}$ ) in the asymptotic expansion of the right-hand side of (2.9) in even (respectively odd) dimension. On focusing for definiteness on the even  $d$  case, we now exploit the identity

$$J_{\nu-1}(k) = J'_\nu(k) + \frac{\nu}{k} J_\nu(k), \quad (2.10)$$

and obtain

$$J_{\nu-1}^2(ix) - e^{2\theta} J_\nu^2(ix) = J_\nu'^2(ix) - \left(\frac{\nu^2}{x^2} + e^{2\theta}\right) J_\nu^2(ix) + \frac{2\nu}{ix} J_\nu(ix) J'_\nu(ix). \quad (2.11)$$

Thus, on defining  $\alpha_\nu \equiv \sqrt{\nu^2 + x^2}$  and using the uniform asymptotic expansions of  $J_\nu(ix)$  and  $J'_\nu(ix)$  summarized in the Appendix we find

$$J_{\nu-1}^2(ix) - e^{2\theta} J_\nu^2(ix) \sim \frac{(ix)^{2(\nu-1)}}{2\pi} \alpha_\nu e^{2\alpha_\nu} e^{-2\nu \log(\nu + \alpha_\nu)} \left[ \Sigma_1^2 A_\theta(t) + \Sigma_2^2 + 2t \Sigma_1 \Sigma_2 \right], \quad (2.12)$$

where we have defined

$$t \equiv \frac{\nu}{\alpha_\nu}, \quad (2.13)$$

$$A_\theta(t) \equiv 1 + (t^2 - 1)(1 - e^{2\theta}). \quad (2.14)$$

As expected, our formulae reduce, at  $\theta = 0$ , to the asymptotic expansions used in Ref. [12]. From now on we need to recall that the functions  $\Sigma_1$  and  $\Sigma_2$  have asymptotic series in the form

$$\Sigma_1 \sim \sum_{k=0}^{\infty} \frac{u_k(t)}{\nu^k}, \quad (2.15)$$

$$\Sigma_2 \sim \sum_{k=0}^{\infty} \frac{v_k(t)}{\nu^k}, \quad (2.16)$$

where  $u_k$  and  $v_k$  are the Debye polynomials given in Ref. [24]. The asymptotic expansions on the right-hand sides of (2.15) and (2.16) can be re-expressed as

$$\sum_{k=0}^{\infty} \frac{u_k(t)}{\nu^k} \sim \sum_{j=0}^{\infty} \frac{a_j(t)}{(\alpha_\nu)^j}, \quad (2.17)$$

$$\sum_{k=0}^{\infty} \frac{v_k(t)}{\nu^k} \sim \sum_{j=0}^{\infty} \frac{b_j(t)}{(\alpha_\nu)^j}, \quad (2.18)$$

where

$$a_i(t) = \frac{u_i(t)}{t^i}, \quad b_i(t) = \frac{v_i(t)}{t^i}, \quad \forall i \geq 0. \quad (2.19)$$

Now the asymptotic expansion (2.12) suggests defining

$$\tilde{\Sigma} \equiv \Sigma_1^2 A_\theta(t) + \Sigma_2^2 + 2t\Sigma_1\Sigma_2, \quad (2.20)$$

and hence studying the asymptotic expansion of  $\log(\tilde{\Sigma})$  in the relation to be used in (2.9), i.e.

$$\begin{aligned} & \log \left[ (ix)^{-2(\nu-1)} (J_{\nu-1}^2 - e^{2\theta} J_\nu^2)(ix) \right] \\ & \sim -\log(2\pi) + \log \alpha_\nu + 2\alpha_\nu - 2\nu \log(\nu + \alpha_\nu) + \log \tilde{\Sigma}. \end{aligned} \quad (2.21)$$

From the relations (2.13)–(2.20)  $\tilde{\Sigma}$  has the asymptotic expansion

$$\tilde{\Sigma} \sim \sum_{p=0}^{\infty} \frac{c_p}{(\alpha_\nu)^p}, \quad (2.22)$$



where the first few  $c_p$  coefficients read

$$c_0 = A_\theta + 1 + 2t, \quad (2.23)$$

$$c_1 = 2a_1 A_\theta + 2b_1 + 2t(a_1 + b_1), \quad (2.24)$$

$$c_2 = (2a_2 + a_1^2)A_\theta + (2b_2 + b_1^2) + 2t(a_2 + b_2 + a_1 b_1), \quad (2.25)$$

$$c_3 = 2(a_3 + a_1 a_2)A_\theta + 2(b_3 + b_1 b_2) + 2t(a_3 + b_3 + a_1 b_2 + a_2 b_1). \quad (2.26)$$

Now defining

$$\Sigma \equiv \frac{\tilde{\Sigma}}{c_0}, \quad (2.27)$$

and making the usual expansion

$$\log(1 + f) = \sum_{j=0}^{\infty} (-1)^{j+1} \frac{f^j}{j}, \quad (2.28)$$

valid as  $f \rightarrow 0$ , we find

$$\log \tilde{\Sigma} = \log c_0 + \log \Sigma \sim \log c_0 + \sum_{p=1}^{\infty} \frac{A_p}{(\alpha_\nu)^p}, \quad (2.29)$$

where explicit formulae can be given for all  $A_p$ . In particular

$$A_1 = \frac{c_1}{c_0}, \quad (2.30)$$

$$A_2 = \frac{c_2}{c_0} - \frac{1}{2}(A_1)^2, \quad (2.31)$$

$$A_3 = \frac{c_3}{c_0} - A_1 A_2 - \frac{1}{6}(A_1)^3. \quad (2.32)$$

Using the definition

$$f_\theta(t) \equiv 1 + \frac{(t-1)}{2}(1 - e^{2\theta}), \quad (2.33)$$

jointly with our previous formulae and the explicit form of Debye polynomials [24], a lengthy calculation yields as many  $A_p$  terms as are needed. For example (cf. Ref. [12])

$$A_1 = \frac{1}{4} - \frac{5}{12}t^2 + \frac{\frac{1}{2}(t^2 - 1)}{f_\theta(t)}, \quad (2.34)$$

$$A_2 = \sum_{k=0}^2 \frac{\Omega_k(t)}{(f_\theta(t))^k}, \quad (2.35)$$

$$A_3 = \sum_{k=0}^3 \frac{\omega_k(t)}{(f_\theta(t))^k}, \quad (2.36)$$

where

$$\Omega_0(t) = \frac{1}{8} - \frac{3}{4}t^2 + \frac{5}{8}t^4, \quad (2.37)$$

$$\Omega_1(t) = -\frac{t}{8} + \frac{5}{8}t^2 + \frac{t^3}{8} - \frac{5}{8}t^4, \quad (2.38)$$

$$\Omega_2(t) = -\frac{1}{8} + \frac{1}{4}t^2 - \frac{1}{8}t^4, \quad (2.39)$$

$$\omega_0(t) = \frac{25}{512} - \frac{531}{320}t^2 + \frac{221}{64}t^4 - \frac{1105}{576}t^6, \quad (2.40)$$

$$\omega_1(t) = -\frac{1}{16} - \frac{t}{16} + \frac{5}{4}t^2 + \frac{3}{8}t^3 - \frac{49}{16}t^4 - \frac{5}{16}t^5 + \frac{15}{8}t^6, \quad (2.41)$$

$$\omega_2(t) = -\frac{t}{16} + \frac{5}{16}t^2 + \frac{t^3}{8} - \frac{5}{8}t^4 - \frac{t^5}{16} + \frac{5}{16}t^6, \quad (2.42)$$

$$\omega_3(t) = -\frac{1}{24} + \frac{t^2}{8} - \frac{t^4}{8} + \frac{t^6}{24}. \quad (2.43)$$

In the calculation, all factors  $1 + t$  in the denominators of  $A_1, A_2$  and  $A_3$  have cancelled against factors in the numerators, as in the  $\theta = 0$  case [12]. Moreover, a simple but non-trivial consistency check shows that, at  $\theta = 0$ , equations (2.34)–(2.43) yield  $A_1, A_2$  and  $A_3$  in agreement with Ref. [12].

As a result of all these formulae we find

$$\log\left[(ix)^{-2(\nu-1)}(J_{\nu-1}^2 - e^{2\theta}J_{\nu}^2)(ix)\right] \sim \sum_{i=1}^{\infty} \tilde{S}_i(\nu, \alpha_{\nu}(x)), \quad (2.44)$$

where the first few functions  $\tilde{S}_i$  read (cf. Ref. [12])

$$\tilde{S}_1 \equiv -\log \pi + 2\alpha_{\nu}, \quad (2.45)$$

$$\tilde{S}_2 \equiv -(2\nu - 1) \log(\nu + \alpha_{\nu}) + \log f_{\theta}(t), \quad (2.46)$$

$$\tilde{S}_3 \equiv \frac{A_1}{\alpha_{\nu}}, \quad (2.47)$$

$$\tilde{S}_4 \equiv \frac{A_2}{\alpha_{\nu}^2}, \quad (2.48)$$

$$\tilde{S}_5 \equiv \frac{A_3}{\alpha_{\nu}^3}. \quad (2.49)$$

Expanding  $\deg(n)$  in powers of  $n$ , the infinite sum over  $n$  in the expression (2.9) can be evaluated with the help of formulae derived using contour integration, i.e. [22]:

$$\sum_{p=0}^{\infty} p^{2k} \alpha_p^{-2k-l} \sim \frac{\Gamma\left(k + \frac{1}{2}\right) \Gamma\left(\frac{l}{2} - \frac{1}{2}\right)}{2\Gamma\left(k + \frac{l}{2}\right)} x^{1-l}, \quad k = 1, 2, \dots, \quad (2.50)$$

$$\sum_{p=0}^{\infty} p \alpha_p^{-1-l} \sim \frac{x^{1-l}}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{2^r}{r!} \tilde{B}_r x^{-r} \frac{\Gamma\left(\frac{r}{2} + \frac{1}{2}\right) \Gamma\left(\frac{l}{2} - \frac{1}{2} + \frac{r}{2}\right)}{2\Gamma\left(\frac{1}{2} + \frac{l}{2}\right)} \cos \frac{r\pi}{2}. \quad (2.51)$$

Here  $l$  is a real number larger than 1 and  $\tilde{B}_0 = 1, \tilde{B}_2 = \frac{1}{6}, \tilde{B}_4 = -\frac{1}{30}$  etc. are Bernoulli numbers. In arbitrary dimension  $d$ , the expansion of  $\deg(n)$  in powers of  $n$  is cumbersome and a systematic formula suitable for all  $d$  is given in Eq. (4.3).

### III. CONFORMAL ANOMALY ON THE 4-BALL

As a first application, we show how the  $\zeta(0)$  calculation of Ref. [12] is extended to our boundary conditions involving  $\theta$ , leading in turn to the eigenvalue conditions (1.6) and

(1.7). Only calculations for (1.7) are presented, but the full  $\zeta(0)$  value, which expresses the conformal anomaly for a massless Dirac spin- $\frac{1}{2}$  field (we do not study the coupling of spinor fields to gauge fields, which would lead us instead to the subject of chiral anomalies), receives a contribution from (1.6) obtained by replacing  $\theta$  with  $-\theta$  in the result from (1.7).

In 4 dimensions, Sec. II shows that  $\zeta(0) = a_2$  is equal to  $\frac{1}{2}$  times the coefficient of  $x^{-6}$  in the asymptotic expansion of the right-hand side of Eq. (2.9) at  $d = 4$ . On setting  $m \equiv n + 2$  the latter reads (here  $\alpha_m(x) \equiv \sqrt{m^2 + x^2}$ )

$$\sum_{m=0}^{\infty} (m^2 - m) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \sum_{i=1}^{\infty} \tilde{S}_i(m, \alpha_m(x)) \sim \sum_{i=1}^{\infty} W_{\infty}^i, \quad (3.1)$$

with  $W_{\infty}^i$  corresponding to the third derivative of  $\tilde{S}_i$ , for all  $i$ . The terms  $W_{\infty}^i$  contribute to  $a_2$  in 4 dimensions only up to  $i = 5$ , and hence only their analysis is presented hereafter.

### *Contribution of $W_{\infty}^1$ and $W_{\infty}^2$*

The term  $W_{\infty}^1$  is given by

$$W_{\infty}^1 = \sum_{m=0}^{\infty} (m^2 - m) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 (-\log(\pi) + 2\alpha_m), \quad (3.2)$$

which is unaffected by  $\theta$ -dependent boundary conditions. Thus, we know from Ref. [12] that Eqs. (2.50) and (2.51) imply vanishing contribution to  $\zeta(0)$ .

The term  $W_{\infty}^2$  reads (here  $t \equiv \frac{m}{\alpha_m}$ )

$$\begin{aligned} W_{\infty}^2 &= \sum_{m=0}^{\infty} (m^2 - m) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \left[ -(2m - 1) \log(m + \alpha_m) + \log f_{\theta}(t) \right] \\ &= W_{\infty}^{2,A} + W_{\infty}^{2,B}, \end{aligned} \quad (3.3)$$

where  $W_{\infty}^{2,A}$  is the  $\theta$ -independent part while  $W_{\infty}^{2,B}$  denotes the part involving  $\log f_{\theta}(t)$ . From Ref. [12] we know that  $W_{\infty}^{2,A}$  contributes

$$\zeta^{2,A}(0) = -\frac{1}{120} + \frac{1}{24} = \frac{1}{30}. \quad (3.4)$$

The  $\log f_{\theta}$  is dealt with by defining

$$\gamma \equiv \frac{1}{2}(1 - e^{2\theta}) = -e^{\theta} \sinh \theta, \quad (3.5)$$

$$\beta \equiv \frac{\gamma}{(1-\gamma)} = -\tanh \theta, \quad (3.6)$$

and hence writing (see (2.33))

$$\log f_\theta = \log(1-\gamma) + \log(1+\beta m \alpha_m^{-1}). \quad (3.7)$$

At this stage we can exploit Eq. (2.28), with  $f \equiv \beta m \alpha_m^{-1} < 1$  since  $\alpha_m$  is always evaluated at large  $x$ , and hence we find

$$W_\infty^{2,B} = \frac{1}{8} \sum_{k=1}^{\infty} (-1)^k (k+2)(k+4) \beta^k \sum_{m=0}^{\infty} (m^{2+k} - m^{1+k}) \alpha_m^{-k-6}, \quad (3.8)$$

where the interchange of the orders of summation is made possible by uniform convergence. Interestingly, this sum contributes infinitely many  $x^{-6}$  terms with equal magnitude and opposite sign, so that  $\zeta(0)$  is unaffected. More precisely, we consider odd and even values of  $k$  and hence define

$$F_1 \equiv \sum_{m=0}^{\infty} m^{2k+3} \alpha_m^{-2k-7}, \quad F_2 \equiv \sum_{m=0}^{\infty} m^{2k+2} \alpha_m^{-2k-7}, \quad k = 0, 1, 2, \dots, \quad (3.9)$$

$$F_3 \equiv \sum_{m=0}^{\infty} m^{2k+2} \alpha_m^{-2k-6}, \quad F_4 \equiv \sum_{m=0}^{\infty} m^{2k+1} \alpha_m^{-2k-6}, \quad k = 1, 2, \dots \quad (3.10)$$

By virtue of (2.51),  $F_1$  contributes to  $x^{-6}$  with zero weight for all  $k$  because of the  $\cos \frac{3\pi}{2}$  coefficient. Moreover,  $F_2$  is proportional to  $x^{-4}$  by virtue of (2.50). The sum  $F_3$  is instead proportional to  $x^{-3}$  (again by (2.50)), while  $F_4$  is such that its contribution  $\delta_{F_4}(x; k)$  to  $\zeta(0)$  reads

$$\delta_{F_4}(x; 1) = -\frac{1}{12} x^{-6} \left( \frac{\Gamma(3)}{\Gamma(3)} - \frac{\Gamma(4)}{\Gamma(4)} \right) = 0, \quad (3.11)$$

$$\delta_{F_4}(x; 2) - \delta_{F_4}(x; 1) = -\frac{1}{12} x^{-6} \left( -\frac{\Gamma(4)}{\Gamma(4)} + \frac{\Gamma(5)}{\Gamma(5)} \right) = 0, \quad (3.12)$$

and infinitely many other relations along the same lines.

*Effect of  $W_\infty^3, W_\infty^4$  and  $W_\infty^5$*

The term  $W_\infty^3$  is equal to

$$\begin{aligned}
W_\infty^3 &= \sum_{m=0}^{\infty} (m^2 - m) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \left[ \frac{1}{4} \alpha_m^{-1} - \frac{5}{12} m^2 \alpha_m^{-3} \right. \\
&\quad \left. + \frac{1}{2} (1 - \gamma)^{-1} \left( \frac{m^2}{\alpha_m^3} - \frac{1}{\alpha_m} \right) (1 + \beta m \alpha_m^{-1})^{-1} \right] \\
&= W_\infty^{3,A} + W_\infty^{3,B} + W_\infty^{3,C} + W_\infty^{3,D},
\end{aligned} \tag{3.13}$$

where  $W_\infty^{3,A}$  and  $W_\infty^{3,B}$  are the first two,  $\theta$ -independent sums, while  $W_\infty^{3,C}$  and  $W_\infty^{3,D}$  are the sums depending on  $\theta$  through  $\gamma$  and  $\beta$ . By virtue of the large- $x$  nature of the whole analysis, we can expand  $(1 + \beta m \alpha_m^{-1})^{-1}$  according to

$$(1 + \beta m \alpha_m^{-1})^{-1} = \sum_{k=0}^{\infty} (-1)^k \beta^k m^k \alpha_m^{-k}. \tag{3.14}$$

Upon exploiting the identity

$$\left( \frac{1}{2x} \frac{d}{dx} \right)^3 \alpha_m^{-l} = -\frac{1}{8} l(l+2)(l+4) \alpha_m^{-l-6}, \tag{3.15}$$

we find that  $W_\infty^{3,A}$  and  $W_\infty^{3,B}$  do not contribute to  $\zeta(0)$  by virtue of (2.50) and (2.51). The same holds for  $W_\infty^{3,C}$  and  $W_\infty^{3,D}$ , but the proof requires more intermediate steps, as follows.

The term  $W_\infty^{3,C}$  is given by

$$W_\infty^{3,C} = -\frac{1}{16} (1 - \gamma)^{-1} \sum_{k=0}^{\infty} (-1)^k \beta^k (k+3)(k+5)(k+7) \sum_{m=0}^{\infty} (m^4 - m^3) m^k \alpha_m^{-k-9}. \tag{3.16}$$

Looking at even and odd values of  $k$ , this suggests defining

$$G_1 \equiv \sum_{m=0}^{\infty} m^{2k+4} \alpha_m^{-2k-9}, \quad G_2 \equiv \sum_{m=0}^{\infty} m^{2k+3} \alpha_m^{-2k-9}, \quad k = 0, 1, 2, \dots, \tag{3.17}$$

$$G_3 \equiv \sum_{m=0}^{\infty} m^{2k+5} \alpha_m^{-2k-10}, \quad G_4 \equiv \sum_{m=0}^{\infty} m^{2k+4} \alpha_m^{-2k-10}, \quad k = 0, 1, 2, \dots. \tag{3.18}$$

Now  $G_1$  and  $G_4$  are proportional to  $x^{-4}$  and  $x^{-5}$  respectively by virtue of (2.50), and hence do not contribute to  $\zeta(0)$ .  $G_2$  contains  $x^{-6}$  weighted by a coefficient proportional to  $\cos \frac{\pi}{2}$ , for all  $k$ , and hence does not contribute to  $\zeta(0)$ . Last,  $G_3$  is such that its contribution  $\delta_{G_3}(x; k)$  to  $\zeta(0)$  reads

$$\delta_{G_3}(x; 0) = -\frac{1}{12} x^{-6} \left( \frac{\Gamma(5/2)}{\Gamma(5/2)} - 2 \frac{\Gamma(4)}{\Gamma(4)} + \frac{\Gamma(5)}{\Gamma(5)} \right) = 0, \tag{3.19}$$

jointly with infinitely many other relations along the same lines.

The term  $W_\infty^{3,D}$  is given by

$$W_\infty^{3,D} = \frac{1}{16}(1-\gamma)^{-1} \sum_{k=0}^{\infty} (-1)^k \beta^k (k+1)(k+3)(k+5) \sum_{m=0}^{\infty} (m^{2+k} - m^{1+k}) \alpha_m^{-k-7}. \quad (3.20)$$

Here, too, we split the sum over  $k$  into sums over all even and odd values of  $k$ . We find therefore, exploiting (2.50) and (2.51), either contributions proportional to  $x^{-4}$  and  $x^{-5}$ , or  $x^{-6}$  terms weighted by  $\cos \frac{\pi}{2}$ , or the contributions resulting from

$$H_3 \equiv \sum_{m=0}^{\infty} m^{2k+3} \alpha_m^{-2k-8}, \quad k = 0, 1, 2, \dots, \quad (3.21)$$

which occur with opposite signs for all  $k$ .

The general formula for  $W_\infty^4$  reads

$$\begin{aligned} W_\infty^4 &= \sum_{m=0}^{\infty} (m^2 - m) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \frac{A_2}{\alpha_m^2} \\ &= \sum_{m=0}^{\infty} (m^2 - m) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \left\{ \alpha_m^{-2} \left[ \sum_{r=0}^2 j_{0,r} m^{2r} \alpha_m^{-2r} \right. \right. \\ &\quad \left. \left. + f_\theta^{-1} \sum_{r=1}^4 j_{1,r} m^r \alpha_m^{-r} + f_\theta^{-2} \sum_{r=0}^2 j_{2,r} m^{2r} \alpha_m^{-2r} \right] \right\} \\ &= W_\infty^{4,A} + W_\infty^{4,B} + W_\infty^{4,C}, \end{aligned} \quad (3.22)$$

where  $j_{0,r}$ ,  $j_{1,r}$  and  $j_{2,r}$  are the coefficients in the polynomials  $\Omega_0$ ,  $\Omega_1$  and  $\Omega_2$  respectively (see (2.37)–(2.39)), and negative powers of  $f_\theta$  are expanded by exploiting

$$(1+f)^{-s} = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+s)}{k! \Gamma(s)} f^k \text{ as } f \rightarrow 0. \quad (3.23)$$

Since

$$W_\infty^{4,A} = -\frac{1}{8} \sum_{r=0}^2 j_{0,r} (2r+2)(2r+4)(2r+6) \sum_{m=0}^{\infty} (m^{2r+2} - m^{2r+1}) \alpha_m^{-2r-8}, \quad (3.24)$$

the basic formulae (2.50) and (2.51) imply a contribution to  $\zeta(0)$  equal to

$$\frac{1}{2} \sum_{r=0}^2 j_{0,r} = 0. \quad (3.25)$$

Moreover, since

$$\begin{aligned}
W_\infty^{4,B} &= -\frac{1}{8}(1-\gamma)^{-1} \sum_{r=1}^4 j_{1,r} \sum_{k=0}^{\infty} (-1)^k \beta^k (k+r+2)(k+r+4)(k+r+6) \\
&\quad \times \sum_{m=0}^{\infty} (m^2 - m) m^{k+r} \alpha_m^{-k-r-8},
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
W_\infty^{4,C} &= -\frac{1}{8}(1-\gamma)^{-2} \sum_{r=0}^4 j_{2,r} \sum_{k=0}^{\infty} (-1)^k (k+1) \beta^k (k+2r+2)(k+2r+4)(k+2r+6) \\
&\quad \times \sum_{m=0}^{\infty} (m^{2(r+1)+k} - m^{2r+k+1}) \alpha_m^{-k-2r-8},
\end{aligned} \tag{3.27}$$

repeated application of (2.50) and (2.51) yields contributions to  $\zeta(0)$  equal to

$$\frac{1}{2}(1-\gamma)^{-1}(1+\beta)^{-1} \sum_{r=1}^4 j_{1,r} = 0, \tag{3.28}$$

and

$$\frac{1}{2}(1-\gamma)^{-2} \left( \sum_{k=0}^{\infty} (-1)^k (k+1) \beta^k \right) \sum_{r=0}^2 j_{2,r} = 0, \tag{3.29}$$

respectively. The results (3.25), (3.28) and (3.29) are all vanishing because of the peculiar properties of the  $j_{0,r}$ ,  $j_{1,r}$  and  $j_{2,r}$  coefficients.

Last, the general formula for  $W_\infty^5$  reads

$$\begin{aligned}
W_\infty^5 &= \sum_{m=0}^{\infty} (m^2 - m) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \frac{A_3}{\alpha_m^3} \\
&= \sum_{m=0}^{\infty} (m^2 - m) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \left\{ \alpha_m^{-3} \left[ \sum_{r=0}^3 \sigma_{0,2r} m^{2r} \alpha_m^{-2r} \right. \right. \\
&\quad + f_\theta^{-1} \sum_{r=0}^6 \sigma_{1,r} m^r \alpha_m^{-r} + f_\theta^{-2} \sum_{r=1}^6 \sigma_{2,r} m^r \alpha_m^{-r} \\
&\quad \left. \left. + f_\theta^{-3} \sum_{r=0}^3 \sigma_{3,2r} m^{2r} \alpha_m^{-2r} \right] \right\} \\
&= W_\infty^{5,A} + W_\infty^{5,B} + W_\infty^{5,C} + W_\infty^{5,D},
\end{aligned} \tag{3.30}$$

where  $\sigma_{0,2r}$ ,  $\sigma_{1,r}$ ,  $\sigma_{2,r}$  and  $\sigma_{3,2r}$  are the coefficients in the polynomials  $\omega_0$ ,  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  respectively (see (2.40)–(2.43)), and also  $f_\theta^{-3}$  is expanded by exploiting (3.23). Now the term  $W_\infty^{5,A}$ , which is the  $\theta$ -independent part of (3.30), yields a non-vanishing contribution to  $\zeta(0)$  equal to



$$\frac{1}{2} \left( - \sum_{r=0}^3 \sigma_{0,2r} \right) = -\frac{1}{360}, \quad (3.31)$$

while the terms  $W_{\infty}^{5,B}$ ,  $W_{\infty}^{5,C}$  and  $W_{\infty}^{5,D}$ , resulting from  $f_{\theta}^{-1}$ ,  $f_{\theta}^{-2}$  and  $f_{\theta}^{-3}$  respectively, give vanishing contribution obtained as follows:

$$\frac{1}{2} (1 - \gamma)^{-1} (1 + \beta)^{-1} \left( - \sum_{r=0}^6 \sigma_{1,r} \right) = 0 \text{ from } W_{\infty}^{5,B}, \quad (3.32)$$

$$\frac{1}{2} (1 - \gamma)^{-2} \left( \sum_{k=0}^{\infty} (-1)^k (k+1) \beta^k \right) \left( - \sum_{r=1}^6 \sigma_{2,r} \right) = 0 \text{ from } W_{\infty}^{5,C}, \quad (3.33)$$

$$\frac{1}{2} (1 - \gamma)^{-3} \left( \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)(k+2)}{2} \beta^k \right) \left( - \sum_{r=0}^3 \sigma_{3,2r} \right) = 0 \text{ from } W_{\infty}^{5,D}, \quad (3.34)$$

by repeated application of (2.50) and (2.51).

By virtue of (3.4), (3.25), (3.28), (3.29), (3.31)–(3.34) we find

$$\zeta(0) = 2 \left( \frac{1}{30} - \frac{1}{360} \right) = \frac{11}{180}, \quad (3.35)$$

for a massless Dirac field on the 4-ball, bearing in mind that also the eigenvalue condition (1.6) should be considered. Interestingly, such a  $\zeta(0)$  value in 4 dimensions is independent of  $\theta$ , and agrees with the result in Ref. [12], where a massless spin- $\frac{1}{2}$  field with half as many components as a Dirac field was instead considered.

The proof of vanishing contributions to  $\zeta(0)$  from the infinite sums  $F_4$ ,  $G_3$  and  $H_3$  can be made more systematic and elegant by remarking that a recursive scheme exists for which

$$H_3 = \left( 1 + \frac{x}{2(k+3)} \frac{d}{dx} \right) F_4, \quad (3.36)$$

$$G_3 = \left( 1 + \frac{x}{2(k+4)} \frac{d}{dx} \right) H_3, \quad (3.37)$$

so that one only needs to look at  $F_4$  (see (3.10)), which can be evaluated exactly as a function of  $x$  for all  $k$  by exploiting the Euler–Maclaurin formula [3,12].

#### IV. HEAT-KERNEL COEFFICIENTS IN GENERAL DIMENSION $D$

Our aim in this section is to apply the formalism in such a way that in principle all heat-kernel coefficients in any dimension  $d$  can be obtained. Therefore we will need the large- $x$  expansion of (see (2.21) and (3.5))

$$\left(\frac{1}{2x}\frac{d}{dx}\right)^{1+d/2} \left[ 2\alpha_\nu - (2\nu - 1) \log(\nu + \alpha_\nu) + \log(1 + \gamma(t - 1)) + \sum_{p=1}^{\infty} \frac{A_p}{(\alpha_\nu)^p} \right]. \quad (4.1)$$

The first complication compared to  $d = 4$  is that now, dealing with arbitrary dimension  $d$ , we need an arbitrary number of derivatives. This is easily generalized in some cases, i.e.

$$\left(\frac{1}{2x}\frac{d}{dx}\right)^j \alpha_\nu = (-1)^{j+1} \frac{(2j-3)!!}{2^j} \alpha_\nu^{1-2j},$$

in others at least in the form of a large- $\alpha_\nu$  expansion,

$$\begin{aligned} \left(\frac{1}{2x}\frac{d}{dx}\right)^j \log(\nu + \alpha_\nu) &\sim \frac{1}{2} (-1)^{j+1} \Gamma(j) \alpha_\nu^{-2j} \\ &\quad + \frac{(-1)^j}{2^j} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k+2j-2)!!}{k!!} \nu^k \alpha_\nu^{-k-2j}, \\ \left(\frac{1}{2x}\frac{d}{dx}\right)^j \log(1 + \gamma(t-1)) &\sim \frac{(-1)^j}{2^j} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k+2j-2)!!}{k!!} \nu^k \beta^k \alpha_\nu^{-k-2j}. \end{aligned}$$

As we have seen in Secs. II and III, to deal with the  $A_p$  contributions, we need terms of the type

$$\begin{aligned} \left(\frac{1}{2x}\frac{d}{dx}\right)^j \frac{t^i}{(1 + \gamma(t-1)^l) \alpha_\nu^p} &= \frac{(-1)^j}{2^j} \frac{1}{(1-\gamma)^l} \\ &\times \sum_{u=0}^{\infty} (-1)^u \frac{\Gamma(l+u)}{u! \Gamma(l)} \beta^u \nu^{u+i} \frac{(u+i+p+2j-2)!!}{(u+i+p-2)!!} \alpha_\nu^{-u-i-p-2j}. \end{aligned}$$

The relevant case is  $j = 1 + d/2$  and the contribution of each term to the zeta function is found by summing over  $n$ , taking the degeneracy into account.

Let us now show how the general procedure works in the case of the  $\alpha_\nu$ -term. The contribution to  $\zeta(1 + d/2, x^2)$  is

$$B = (-1)^{d/2} \frac{d_s}{2\Gamma(1 + d/2)} \sum_{n=0}^{\infty} \binom{d+n-2}{n} \left(\frac{1}{2x}\frac{d}{dx}\right)^{1+d/2} 2\alpha_\nu$$

$$\begin{aligned}
&= d_s \frac{(d-1)!!}{2^{1+d/2} \Gamma\left(1 + \frac{d}{2}\right)} \sum_{n=0}^{\infty} \binom{d+n-2}{n} \alpha_{\nu}^{-1-d} \\
&= d_s \frac{(d-1)!!}{2^{1+d/2} \Gamma\left(1 + \frac{d}{2}\right)} \sum_{n=0}^{\infty} \binom{d+n-2}{n} (\nu^2 + x^2)^{-\frac{1+d}{2}}, \tag{4.2}
\end{aligned}$$

of which we need the large- $x$  expansion. A simple expansion in inverse powers of  $x$  is not allowed; instead we employ a Mellin integral representation. As is clear from the above equation, the Barnes zeta function [25–28]

$$\zeta_{\mathcal{B}}(s, a) \equiv \sum_{n=0}^{\infty} \binom{d+n-2}{n} (n+a)^{-s} = \sum_{\vec{m}=0}^{\infty} (a + m_1 + \dots + m_{d-1})^{-s},$$

will play a crucial role. We need to separate the  $\nu$  and  $x$  dependence in (4.2), more generally in expressions of the form

$$\begin{aligned}
C(j, s) &:= \sum_{n=0}^{\infty} \binom{d+n-2}{n} \nu^j \alpha_{\nu}^{-j-s} \\
&= \frac{1}{\Gamma\left(\frac{s+j}{2}\right)} \sum_{n=0}^{\infty} \binom{d+n-2}{n} \nu^j \int_0^{\infty} dt t^{\frac{s+j}{2}-1} e^{-(\nu^2+x^2)t}.
\end{aligned}$$

The  $\nu$  and  $x$  dependence is separated by employing for  $\Re c > 0$ ,

$$e^{-\nu^2 t} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) \nu^{-2\alpha} t^{-\alpha}.$$

For  $\Re s$  large enough we continue

$$\begin{aligned}
C(j, s) &= \frac{1}{\Gamma\left(\frac{s+j}{2}\right)} \sum_{n=0}^{\infty} \binom{d+n-2}{n} \nu^j \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) \nu^{-2\alpha} \int_0^{\infty} dt t^{\frac{s+j}{2}-\alpha-1} e^{-x^2 t} \\
&= \frac{1}{\Gamma\left(\frac{s+j}{2}\right)} \sum_{n=0}^{\infty} \binom{d+n-2}{n} \nu^j \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) \nu^{-2\alpha} \Gamma\left(\frac{s+j}{2} - \alpha\right) x^{2\alpha-s-j}.
\end{aligned}$$

The sum and integral may be interchanged upon choosing  $\Re c > (j+d-1)/2$  and we find

$$C(j, s) = \frac{1}{\Gamma\left(\frac{s+j}{2}\right)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) \Gamma\left(\frac{s+j}{2} - \alpha\right) x^{2\alpha-s-j} \zeta_{\mathcal{B}}\left(2\alpha - j, \frac{d}{2}\right).$$

On shifting the contour to the left we pick up the large- $x$  expansion of  $C(j, s)$ . In order to find  $a_{n/2}$ , we are interested in the term that behaves as  $x^{-n-2}$  and need to evaluate the

residue at  $a = (s + j - n - 2)/2$ . In all cases we encounter, the only relevant pole will come from  $\zeta_{\mathcal{B}}$  and for these cases

$$C(j, s) \sim \frac{\Gamma\left(\frac{s+j-n}{2} - 1\right)}{2\Gamma\left(\frac{s+j}{2}\right)} \Gamma\left(1 + \frac{n}{2}\right) \text{Res } \zeta_{\mathcal{B}}\left(s - n - 2, \frac{d}{2}\right) x^{-n-2} + \text{irrelevant} . \quad (4.3)$$

From here, e.g., one obtains

$$B = \frac{1}{4\sqrt{\pi}} d_s \Gamma\left(\frac{d-1-n}{2}\right) \text{Res } \zeta_{\mathcal{B}}\left(d-1-n, \frac{d}{2}\right) \frac{\Gamma\left(1 + \frac{n}{2}\right)}{\Gamma\left(1 + \frac{d}{2}\right)}.$$

The procedure just outlined can be applied to all terms in (4.1). The following list summarizes for each term on the left the contribution to the heat-kernel coefficient  $a_{n/2}$  on the right ( $\theta$  and  $-\theta$  contributions are summed):

$$\begin{aligned} (2\alpha_\nu - 2\nu \log(\nu + \alpha_\nu)) &\rightarrow \frac{d_s}{2\sqrt{\pi}(d-n)} \Gamma\left(\frac{d-n-1}{2}\right) \text{Res } \zeta_{\mathcal{B}}\left(d-1-n, \frac{d}{2}\right), \\ \log(\nu + \alpha_\nu) &\rightarrow \frac{d_s}{2\sqrt{\pi}(d-n)} \Gamma\left(\frac{d-n+1}{2}\right) \text{Res } \zeta_{\mathcal{B}}\left(d-n, \frac{d}{2}\right), \\ \log(1 + \gamma(t-1)) &\rightarrow \frac{d_s}{4} \Gamma\left(\frac{d-n}{2}\right) (\cosh^{d-n} \theta - 1) \text{Res } \zeta_{\mathcal{B}}\left(d-n, \frac{d}{2}\right), \\ \frac{t^i}{\alpha_\nu^p (1 + \gamma(t-1))^l} &\rightarrow -\frac{d_s}{4e^{l\theta} \cosh^l \theta} \text{Res } \zeta_{\mathcal{B}}\left(d+p-n, \frac{d}{2}\right) \times \\ &\quad \left\{ \frac{\Gamma\left(\frac{d+i+p-n}{2}\right)}{\Gamma\left(\frac{i+p}{2}\right)} {}_3F_2\left(\frac{l+1}{2}, \frac{l}{2}, \frac{d+i+p-n}{2}; \frac{1}{2}, \frac{p+i}{2}; \tanh^2 \theta\right) \right. \\ &\quad \left. + l \tanh \theta \frac{\Gamma\left(\frac{d+i+p+1-n}{2}\right)}{\Gamma\left(\frac{i+p+1}{2}\right)} {}_3F_2\left(\frac{l+1}{2}, 1 + \frac{l}{2}, \frac{d+i+p+1-n}{2}; \frac{3}{2}, \frac{p+i+1}{2}; \tanh^2 \theta\right) \right\} \\ &\quad + \quad \theta \rightarrow -\theta. \end{aligned}$$

For the calculation of heat-kernel coefficients, note that only a finite number of terms contribute. The poles of the Barnes zeta function are located at  $s = 1, \dots, d-1$ , and depending on the values of  $n$  and  $p$  only a finite number of terms needs to be evaluated. In general, to evaluate  $a_{n/2}$ , we need to include all terms up to  $p = n-1$  [12].

The above results resemble very much the structure of the results found for different boundary conditions given in [23,29–32]. In particular, a reduction of the analysis from the ball to the sphere (in form of the Barnes zeta functions) has been achieved. Indeed,

instead of using the presented algorithm we could equally well have used the contour integral method developed in [33,29,30]. The starting point for the zeta function associated with the eigenvalues from (1.7) in this approach reads

$$\zeta_\theta(s) = \sum_{n=0}^{\infty} \deg(n) \int_{\gamma} \frac{dk}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln \left( J_{n+d/2-1}^2(k) - e^{2\theta} J_{n+d/2}^2(k) \right),$$

the contour  $\gamma$  enclosing all eigenvalues of (1.7). One then uses the uniform asymptotic expansion in order to extract the pieces that can contribute to the heat kernel coefficients. Performing the  $k$ -integrals, results analogous to the above are found and final answers, of course, agree.

Given the explicit results in the above list where all ingredients are known, the algorithm can be cast in a form suitable for application of Mathematica. As far as this process is concerned, some remarks are in order. We have presented the results in terms of hypergeometric functions, and as far as we can see keeping  $d, n$  *arbitrary* this is the best one can do. However, as soon as one considers particular values of  $d$  and  $n$ , the hypergeometric function  ${}_3F_2$  “collapses” to  ${}_2F_1$ , which, at the particular values needed, is simply given as an algebraic combination of hyperbolic functions. For example one has

$$\begin{aligned} {}_2F_1(1, 1, 1/2, \tanh^2 \theta) &= \frac{1}{1 - \tanh^2 \theta} + \frac{\tanh \theta \arcsin \tanh \theta}{(1 - \tanh^2 \theta)^{3/2}} \\ &= \cosh^2 \theta (1 + \arcsin \tanh \theta \sinh \theta). \end{aligned}$$

Mathematica will not always replace automatically the hypergeometric functions by this kind of hyperbolic combinations. Since this is essential for further simplifications of final answers, the implementation of some of the Gauss relations is necessary. We have used

$$\begin{aligned} {}_2F_1(\alpha + 1, \beta + 1, \gamma + 1, z) &= \frac{1}{\alpha(1 - z)} \{ \gamma {}_2F_1(\alpha, \beta, \gamma, z) - (\gamma - \alpha) {}_2F_1(\alpha, \beta + 1, \gamma + 1, z) \}, \\ \gamma {}_2F_1(\alpha, \beta, \gamma, z) &= (\gamma - \alpha) {}_2F_1(\alpha, \beta, \gamma + 1, z) + \alpha {}_2F_1(\alpha + 1, \beta, \gamma + 1, z). \end{aligned}$$

These relations guarantee that ultimately all hypergeometric functions are given in very explicit terms and that huge simplifications can be performed explicitly. In Sec. V we have summarized our findings in  $d = 2, 4, 6$  dimensions giving final results up to the coefficient  $a_{d/2}$ .

## V. LIST OF HEAT-KERNEL COEFFICIENTS

We list hereafter the general results we have obtained. The lower coefficients have quite a simple form in all dimensions and the leading three coefficients are as follows:

$$a_0 = \frac{d_s}{2^d \Gamma\left(1 + \frac{d}{2}\right)}, \quad (5.1)$$

$$a_{1/2} = \frac{\sqrt{\pi} d_s}{2^d \Gamma\left(\frac{d}{2}\right)} \left( (\cosh \theta)^{d-1} - 1 \right), \quad (5.2)$$

$$a_1 = \frac{(2d-5)d_s}{3 \cdot 2^d \Gamma\left(\frac{d}{2}\right)} + \frac{d_s}{2^d \Gamma\left(\frac{d}{2}\right)} \left\{ {}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; (\tanh \theta)^2\right) - (d-1) {}_2F_1\left(1, \frac{d+1}{2}; \frac{3}{2}; (\tanh \theta)^2\right) \right\}. \quad (5.3)$$

Moreover, to show the applicability of our algorithms to arbitrary dimensions and in principle to any coefficient we give the following collection of results.

**d=2:**

$$a_0 = \frac{d_s}{4}, \quad (5.4)$$

$$a_{1/2} = \frac{\sqrt{\pi} d_s}{4} (\cosh \theta - 1), \quad (5.5)$$

$$a_1 = -\frac{d_s}{12}. \quad (5.6)$$

**d=4:**

$$a_0 = \frac{d_s}{32}, \quad (5.7)$$

$$a_{1/2} = \frac{\sqrt{\pi} d_s}{16} \left( (\cosh \theta)^3 - 1 \right), \quad (5.8)$$

$$a_1 = -\frac{d_s}{16} \cosh 2\theta, \quad (5.9)$$

$$a_{3/2} = \frac{\sqrt{\pi}d_s}{4096} \left( \operatorname{sech} \left( \frac{\theta}{2} \right) \right)^4 (15 + 20 \cosh \theta - 11 \cosh 2\theta), \quad (5.10)$$

$$a_2 = \frac{11d_s}{720}. \quad (5.11)$$

**d=6:**

$$a_0 = \frac{d_s}{384}, \quad (5.12)$$

$$a_{1/2} = \frac{\sqrt{\pi}d_s}{128} ((\cosh \theta)^5 - 1), \quad (5.13)$$

$$a_1 = -\frac{d_s}{384} (-2 + 6 \cosh 2\theta + \cosh 4\theta), \quad (5.14)$$

$$a_{3/2} = \frac{\sqrt{\pi}d_s}{98304} \left( \operatorname{sech} \left( \frac{\theta}{2} \right) \right)^4 (153 + 212 \cosh \theta + 35 \cosh 2\theta - 32 \cosh 4\theta - 8 \cosh 5\theta), \quad (5.15)$$

$$a_2 = \frac{d_s}{96} \cosh 2\theta, \quad (5.16)$$

$$a_{5/2} = -\frac{\sqrt{\pi}d_s}{805306368} \left( \operatorname{sech} \left( \frac{\theta}{2} \right) \right)^{10} (311902 + 495474 \cosh \theta + 172792 \cosh 2\theta \\ + 14845 \cosh 3\theta - 21590 \cosh 4\theta - 2159 \cosh 5\theta), \quad (5.17)$$

$$a_3 = -\frac{191d_s}{60480}. \quad (5.18)$$

Of course, the result (5.11) for  $a_2$  in dimension four agrees with Eq. (3.35), upon bearing in mind that  $d_s$  is then equal to 4. For  $\theta = 0$  the results agree with the results found previously in [18,34,30].

## VI. CONCLUDING REMARKS

Motivated by quantum cosmology and the problems of quark confinement, we have studied heat-kernel asymptotics for the squared Dirac operator on the Euclidean ball, with local

boundary conditions (1.1) leading to the eigenvalue conditions (1.6) and (1.7). We have first proved that on the 4-ball the  $\zeta(0)$  value is  $\theta$ -independent. Furthermore, arbitrary values of  $d$  have been considered, and several explicit formulae for heat-kernel coefficients in dimension  $d = 2, 4, 6$  have been obtained in Secs. IV and V. Interestingly,  $a_{d/2}$  is always  $\theta$ -independent, while several other heat-kernel coefficients depend on  $\theta$  through hyperbolic functions and their integer powers.

As far as we can see, the key task is now the analysis of heat-kernel asymptotics with local boundary conditions (1.1) on general Riemannian manifolds  $(M, g)$  with boundary  $\partial M$ . One has then to consider the smooth function  $f \in C^\infty(M)$  mentioned after Eq. (1.13), which is replaced by

$$a_{n/2}(f, P, \mathcal{B}) = c_{n/2}(f, P) + b_{n/2}(f, P, \mathcal{B}). \quad (6.1)$$

The interior part  $c_{n/2}$  vanishes for all odd values of  $n$ , whereas the boundary part only vanishes if  $n = 0$ . The interior part is obtained by integrating over  $M$  a linear combination of local invariants of the appropriate dimension, where the coefficients of the linear combination are *universal constants*, independent of  $d$ . Moreover, the boundary part  $b_{n/2}$  is obtained upon integration over  $\partial M$  of another linear combination of local invariants. In that case, however, the structure group is  $O(d-1)$ , and the coefficients of linear combination will depend on  $d$  and  $\theta$  [20] and so they will be universal functions, as it happens if the boundary operator involves tangential derivatives [31,35–37]. This is indeed the case for the boundary condition (1.1). To see this define

$$\chi \equiv ie^{\theta\gamma^5}\gamma^5\gamma_m$$

and introduce the “projections”

$$\Pi_\pm = \frac{1}{2}(1 \pm \chi).$$

In the bulk of our article we considered the operator  $P$  with domain

$$\text{domain}(P) = \{\psi \in C^\infty(V) : \Pi_- \psi|_{\partial M} \oplus \Pi_- D\psi|_{\partial M} = 0\}.$$



We calculate that

$$\Pi_- D\psi \Big|_{\partial M} = \left( \Pi_+^* \nabla_m + \gamma_m \gamma_a \Pi_-^* \nabla_a \right) \Pi_+ \psi \Big|_{\partial M},$$

$a$  being a tangential index, which for Hermitean  $\Pi_\pm$  ( $\theta = 0$ ) would reduce to standard mixed boundary conditions. However, as is easily seen, this is not the case for  $\theta \neq 0$  and tangential derivatives occur in the boundary conditions such that the boundary conditions considered could be termed of mixed oblique type. It is thus expected, that the general form of  $a_{n/2}$  contains all possible local invariants built from  $f$ , Riemann curvature  $R_{bcd}^a$  of  $M$ , bundle curvature  $\Omega_{ab}$  (in case a gauge theory, with vector bundle over  $M$ , is studied), extrinsic curvature  $K_{ij}$  of  $\partial M$ , endomorphism  $E$  (i.e. potential term) coming from the differential operator  $P$ , combinations of  $\gamma$ -matrices coming from the boundary operator, and the covariant derivatives of all these geometric objects, eventually integrating their linear combinations over  $M$  and  $\partial M$  [5,8,9,38]. All these local invariants are multiplied by universal functions which might depend on  $d$  and  $\theta$ . As a next step, the presented special case calculation together with various other ingredients such as conformal transformations [39], index theory [40], redefinition of the covariant derivative [41] will serve to find results valid for arbitrary Riemannian manifolds and bundle curvatures.

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## APPENDIX

The function  $\mathcal{F}$  on the left-hand side of Eq. (1.7) is the product of the entire functions (i.e. functions analytic in the whole complex plane)

$$\mathcal{F}_1 \equiv J_{n+d/2-1} - e^\theta J_{n+d/2} \quad \text{and} \quad \mathcal{F}_2 \equiv J_{n+d/2-1} + e^\theta J_{n+d/2},$$

which can be written in the form

$$\mathcal{F}_1(k) = \gamma_1 k^{n+d/2-1} e^{g_1(k)} \prod_{i=1}^{\infty} \left(1 - \frac{k}{\mu_i}\right) e^{\frac{k}{\mu_i}}, \quad (\text{A1})$$

$$\mathcal{F}_2(k) = \gamma_2 k^{n+d/2-1} e^{g_2(k)} \prod_{i=1}^{\infty} \left(1 - \frac{k}{\nu_i}\right) e^{\frac{k}{\nu_i}}. \quad (\text{A2})$$

In Eqs. (A1) and (A2),  $\gamma_1$  and  $\gamma_2$  are constants,  $g_1$  and  $g_2$  are entire functions, the  $\mu_i$  are the zeros of  $\mathcal{F}_1$  and the  $\nu_i$  are the zeros of  $\mathcal{F}_2$ . The general theory described in Ref. [42] tells us that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are entire functions whose canonical product has genus 1. In other words, by virtue of the asymptotic behaviour of the eigenvalues, one finds that

$$\sum_{i=1}^{\infty} \frac{1}{|\mu_i|} = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{|\nu_i|} = \infty,$$

whereas  $\sum_{i=1}^{\infty} \frac{1}{|\mu_i|^2}$  and  $\sum_{i=1}^{\infty} \frac{1}{|\nu_i|^2}$  are convergent. This is why the exponentials  $e^{\frac{k}{\mu_i}}$  and  $e^{\frac{k}{\nu_i}}$  must appear in Eqs. (A1) and (A2), which are called the canonical-product representations of  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . The genus of the canonical product for  $\mathcal{F}_1$  is the minimum integer  $h$  such that  $\sum_{i=1}^{\infty} \frac{1}{|\mu_i|^{h+1}}$  converges, and similarly for  $\mathcal{F}_2$ , replacing  $\mu_i$  with  $\nu_i$ . If the genus is equal to 1, this ensures that no higher powers of  $\frac{k}{\mu_i}$  and  $\frac{k}{\nu_i}$  are needed in the argument of the exponential. Moreover, even for non-vanishing values of  $\theta$ , it remains true that the zeros of  $\mathcal{F}_1$  are minus the zeros of  $\mathcal{F}_2$ :  $\mu_i = -\nu_i$ , for all  $i$  [12]. Hence one finds eventually

$$\mathcal{F}(k) = \tilde{\gamma} k^{2(n+d/2-1)} \prod_{i=1}^{\infty} \left(1 - \frac{k^2}{\mu_i^2}\right), \quad (\text{A3})$$

where  $\tilde{\gamma} \equiv \gamma_1 \gamma_2$ ,  $\mu_i^2$  are the positive zeros of  $\mathcal{F}(k)$ , and the sum  $(g_1 + g_2)(k)$  can be shown to vanish exactly as in Sec. IV of Ref. [12].

In our paper we use uniform asymptotic expansions of regular Bessel functions  $J_\nu$  and their first derivatives  $J'_\nu$ . On making the analytic continuation  $x \rightarrow ix$  and then defining  $\alpha_\nu \equiv \sqrt{\nu^2 + x^2}$ , one can write

$$J_\nu(ix) \sim \frac{(ix)^\nu}{\sqrt{2\pi}} \alpha_\nu^{-1/2} e^{\alpha_\nu} e^{-\nu \log(\nu + \alpha_\nu)} \Sigma_1, \quad (\text{A4})$$

$$J'_\nu(ix) \sim \frac{(ix)^{\nu-1}}{\sqrt{2\pi}} \alpha_\nu^{1/2} e^{\alpha_\nu} e^{-\nu \log(\nu + \alpha_\nu)} \Sigma_2, \quad (\text{A5})$$

where the functions  $\Sigma_1$  and  $\Sigma_2$  admit the asymptotic expansions

$$\Sigma_1 \sim \sum_{k=0}^{\infty} u_k(\nu/\alpha_\nu)/\nu^k, \quad \Sigma_2 \sim \sum_{k=0}^{\infty} v_k(\nu/\alpha_\nu)/\nu^k,$$

valid uniformly in the order  $\nu$  as  $|x| \rightarrow \infty$ . The functions  $u_k$  and  $v_k$  are polynomials, given by Eqs. (9.3.9) and (9.3.13) on page 366 of Ref. [24].

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